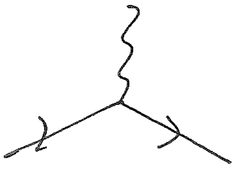
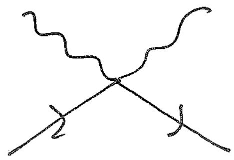
1a) $\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi$

$$\mathcal{L}_I = \underbrace{-ieA_\mu \left(\varphi^\dagger \partial^\mu \varphi - \partial^\mu \varphi^\dagger \varphi\right)}_{(\mathrm{I})} + \underbrace{e^2 A_\mu A^\mu \varphi^\dagger \varphi}_{(\mathrm{II})}$$



(I)



(II)

1b) $$Z[J, J^\dagger, J_\mu] = \int \mathcal{D}A_\mu \int \mathcal{D}\varphi \int \mathcal{D}\varphi^\dagger \, e^{i\int d^4x \left\{ \mathcal{L}_0 + \mathcal{L}_I + J^\dagger \varphi + \varphi^\dagger J + J_\mu A^\mu \right\}}$$

$$= \int \mathcal{D}A_\mu \int \mathcal{D}\varphi \int \mathcal{D}\varphi^\dagger \, e^{i\int d^4x \left\{ \mathcal{L}_0 + J^\dagger \varphi + \varphi^\dagger J + J_\mu A^\mu \right\}} \cdot \left(1 + i\int d^4w \, \mathcal{L}_I(w) + \frac{1}{2!} i^2 \int d^4w_1 \int d^4w_2 \, \mathcal{L}_I(w_1) \mathcal{L}_I(w_2) + \ldots\right)$$

Note that

$$\varphi(w) \, e^{i\int d^4x \{\mathcal{L} + J^\dagger \varphi\}} = \frac{\delta}{\delta i J^\dagger(w)} e^{i\int d^4x \{\mathcal{L} + J^\dagger \varphi\}}$$

$$\varphi^\dagger(w) \, e^{i\int d^4x \{\mathcal{L} + \varphi^\dagger J\}} = \frac{\delta}{\delta i J(w)} e^{i\int d^4x \{\mathcal{L} + \varphi^\dagger J\}}$$

$$A_\mu(w) \cdot e^{i\int d^4x \{\mathcal{L} + J_\mu A^\mu\}} = \frac{\delta}{\delta i J^\mu(w)} e^{i\int d^4x \{\mathcal{L} + J_\mu A^\mu\}}$$

to rewrite $Z$ as follows:

$$Z[J, J^\dagger, J_\mu] = e^{i\int d^4w \, \mathcal{L}_I\left(\frac{\delta}{\delta i J_\mu(w)}, \frac{\delta}{\delta i J^\dagger(w)}, \frac{\delta}{\delta i J(w)}\right)} Z_0[J, J^\dagger, J_\mu]$$

where

$$Z_0[J, J^\dagger, J_\mu] = Z_0[0,0,0] \, e^{-\frac{i}{2}\int d^4x \int d^4y \, J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(y)} \, e^{-i\int d^4x \int d^4y \, J^\dagger(x) D(x-y) J(y)}$$

$$\mathcal{L}_I\left(\frac{\delta}{\delta i J_\mu}, \frac{\delta}{\delta i J^\dagger}, \frac{\delta}{\delta i J}\right) = -ie \frac{\delta}{\delta i J_\mu}\left(\frac{\delta}{\delta i J} \partial_\mu \frac{\delta}{\delta i J^\dagger} - \qquad \right) + e^2 \left(\frac{\delta}{\delta i J_\mu}\right)^2 \frac{\delta}{\delta i J^\dagger} \frac{\delta}{\delta i J}$$

2) $\bar{\Psi}' = \psi^{+\prime}\gamma^0 = \left(e^{\frac{-i\alpha\gamma_5}{2}}\psi\right)^+\gamma^0 = \psi^+ e^{\frac{i\alpha\gamma_5}{2}}\gamma^0 = \bar{\Psi}\gamma^0 e^{\frac{i\alpha\gamma_5}{2}}\gamma^0$

$= \bar{\Psi} e^{\frac{-i\alpha\gamma_5}{2}}$

where we used in order $\gamma_5^+ = \gamma_5$, $\gamma^{0\,2} = \mathbb{1}$, $\{\gamma_5, \gamma^0\} = 0$ and

$$\gamma^0 e^{\frac{i\alpha\gamma_5}{2}}\gamma^0 = \gamma^0 \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\alpha}{2}\right)^n \gamma_5^n\gamma^0 = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{i\alpha}{2}\right)^{2n} - \gamma_5\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{2n+1}$$

$$= e^{-i\alpha\gamma_5}$$

or in inifinitesimal form $= \gamma^0\left(1 + \frac{i\alpha\gamma_5}{2} + O(\alpha^2)\right)\gamma^0 =$

$$= 1 - \frac{i\alpha\gamma_5}{2} + O(\alpha^2) = e^{\frac{-i\alpha\gamma_5}{2}}$$

We show that

$$\mathcal{L}(\varphi_1', \varphi_2', \psi', \bar{\Psi}') = \mathcal{L}(\varphi_1, \varphi_2, \psi, \bar{\Psi})$$

The transformation $(\varphi_1, \varphi_2) \to (\varphi_1', \varphi_2')$ is a rotation in the $(\varphi_1, \varphi_2)$ plane ($SO(2)$).
The scalar part of $\mathcal{L}$ is manifestly invariant under $SO(2)$ because it only depends on the invariant $\varphi_1^2 + \varphi_2^2$ (Show it explicitly)
The Dirac term is invariant under a chiral transformation

$$(\bar{\Psi} i\partial\!\!\!/ \psi)' = \bar{\Psi} e^{\frac{-i\alpha\gamma_5}{2}} i\partial\!\!\!/ e^{\frac{-i\alpha\gamma_5}{2}}\psi = \bar{\psi} i\partial\!\!\!/ \psi \qquad \{\gamma^\mu, \gamma_5\} = 0$$

$$+g\bar{\Psi}'(\varphi_1' + i\gamma_5\varphi_2')\psi' = g\bar{\Psi} e^{\frac{-i\alpha\gamma_5}{2}}(\cos\alpha + i\gamma_5\sin\alpha)(\varphi_1 + i\gamma_5\varphi_2)e^{\frac{-i\alpha\gamma_5}{2}}\psi$$

Then use (*) $\cos\alpha + i\gamma_5\sin\alpha = e^{i\alpha\gamma_5}$ to obtain

$$= g\bar{\Psi} e^{-i\alpha\gamma_5} e^{i\alpha\gamma_5} e^{\frac{-i\alpha\gamma_5}{2}}(\varphi_1 + i\gamma_5\varphi_2)\psi = g\bar{\psi}(\varphi_1 + i\gamma_5\varphi_2)\psi$$

Thus $\mathcal{L}$ is invariant.
(*) Show it by using the Hint.

3) $\partial_\mu J_5^\mu = \partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi) = \bar\psi\overleftarrow{\partial\!\!\!/}\gamma_5\psi + \bar\psi\overrightarrow{\partial\!\!\!/}\gamma_5\psi$

$$= \bar\psi\overleftarrow{\partial\!\!\!/}\gamma_5\psi - \bar\psi\gamma_5\overrightarrow{\partial\!\!\!/}\psi$$

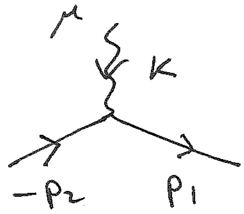
On the EoM one obtains

$$\partial_\mu J_5^\mu = i\bar\psi(eA\!\!\!/+m)\gamma_5\psi + i\bar\psi\gamma_5(eA\!\!\!/+m)\psi$$

$$= ie\bar\psi\{A\!\!\!/,\gamma_5\}\psi + 2im\bar\psi\gamma_5\psi$$

$$= 2im\bar\psi\gamma_5\psi$$

since $\{A\!\!\!/,\gamma_5\} = 0$

4) 


$k = p_1 + p_2$ Feynman rule is $ig_w\gamma^\mu(1-\gamma_5)$

$$\mathcal{A} = ig_w\,\bar u_r(p_1)\gamma^\mu(1-\gamma_5)v_{r'}(p_2)\,\varepsilon^s_\mu(k)$$

$$\mathcal{A}^\dagger = -ig_w\,\bar v_{r'}(p_2)\gamma^0(\gamma^\nu(1-\gamma_5))^\dagger\gamma^0 u_r(p_1)\,\varepsilon^s_\nu(k) \qquad (\varepsilon\text{'s real})$$

$$= -ig_w\,\bar v_{r'}(p_2)\gamma^\nu(1-\gamma_5)u_r(p_1)\,\varepsilon^s_\nu(k)$$

NB $\gamma^0(\gamma^\nu(1-\gamma_5))^\dagger\gamma^0 = \gamma^0(1-\gamma_5)\gamma^{\nu\dagger}\gamma^0 = (1+\gamma_5)\gamma^0\gamma^{\nu\dagger}\gamma^0 = (1+\gamma_5)\gamma^\nu = \gamma^\nu(1-\gamma_5)$

$$X = \frac{1}{3}g_w^2\sum_{r,r'=1,2}\left(\bar u_r(p_1)\gamma^\mu(1-\gamma_5)v_{r'}(p_2)\right)\left(\bar v_{r'}(p_2)\gamma^\nu(1-\gamma_5)u_r(p_1)\right)\times$$

$$\times\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

$$= \frac{1}{3}g_w^2\,\mathrm{Tr}\left[p\!\!\!/_1\gamma^\mu(1-\gamma_5)p\!\!\!/_2\gamma^\nu(1-\gamma_5)\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

$$= \frac{2}{3} g_W^2 \; Tr\left[(1-\gamma_5)\not{p}_2 \gamma^\nu \not{p}_1 \gamma^\mu\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

NB $\quad (1-\gamma_5)^2 = 2(1-\gamma_5) \qquad (1+\gamma_5)^2 = 2(1+\gamma_5)$

The terms with $\gamma_5$ do not contribute because $Tr[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] \propto \varepsilon^{\mu\nu\alpha\beta}$ is totally antisymmetric and it is contracted with the totally symmetric $\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$

you can also show it explicitly:

$$Tr[\gamma_5 \not{p}_2 \gamma^\nu \not{p}_1 \gamma^\mu]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right) =$$

$$= -Tr[\gamma_5 \not{p}_2 \gamma^\mu \not{p}_1 \gamma_\mu] + \frac{1}{M^2} Tr[\gamma_5 \not{p}_2 \not{k} \not{p}_1 \not{k}]$$

$$= 2\, Tr[\gamma_5 \not{p}_2 \not{p}_1] + \frac{1}{M^2} Tr[\gamma_5 \not{p}_2 (2 p_1\cdot k \not{k} - k^2 \not{p}_1)]$$

$$= 0$$

where we used $\quad \gamma^\mu \not{p}_1 \gamma_\mu = -2\not{p}_1$

$$\not{k} \not{p}_1 \not{k} = 2 p_1\cdot k \not{k} - k^2 \not{p}_1$$

and $Tr[\gamma_5 \gamma^\mu \gamma^\nu] = 0$

$$X = \frac{2}{3} g_W^2 \; Tr[\not{p}_2 \gamma^\nu \not{p}_1 \gamma^\mu]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right) =$$

$$= \frac{2}{3} g_W^2 \left\{ -Tr[\not{p}_2 \gamma^\mu \not{p}_1 \gamma_\mu] + \frac{1}{M^2} Tr[\not{p}_2 \not{k} \not{p}_1 \not{k}] \right\}$$

$$= \frac{2}{3} g_W^2 \cdot 4 \left\{ 2 p_1\cdot p_2 + \frac{2}{M^2}(p_1\cdot k)(p_2\cdot k) - \frac{1}{M^2} M^2 p_1\cdot p_2 \right\}$$

$$= \frac{8}{3} g_W^2 \left\{ p_1\cdot p_2 + \frac{2}{M^2}(p_1 k)(p_2 k) \right\}$$

$K = p_1 + p_2$ $\qquad K = (M, \vec{0})$ $\qquad p_1 = (E, \vec{p})$ $\quad p_2 = (E, -\vec{p})$

$p_1^2 = p_2^2 = 0$ $\qquad E = |\vec{p}|$

$K^2 = (p_1 + p_2)^2$ implies $2 p_1 \cdot p_2 = M^2$ thus $p_1 \cdot p_2 = \frac{M^2}{2}$

Energy conservation implies $E = \frac{M}{2}$ and

$p_1 \cdot K = p_2 \cdot K = EM = \frac{M^2}{2}$

Thus $X = \frac{8}{3} g_w^2 M^2$

and

$\Gamma = \frac{1}{16\pi} \frac{1}{M} X = \frac{1}{6\pi} g_w^2 M$